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A Note On Subcontinua of $\beta[0,\infty)-[0,\infty)$

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Abstract. Let $M=\dotplus_{n\in\omega} I_n$ be the topological sum of countably many copies of the unit interval I. For any ultrafilter $u\in\omega^*$, we let $M^u=\bigcap\{cl_{\beta M}(\bigcup\{I_n:n\in A\}):A\in u\}$. It is well-known that M^u is a decomposable continuum with a very nice internal structure (See Mioduszewski[7], Smith[10] and Zhu[11]). In this paper, we show

(1) Every nondegenerate subcontinuum of $\beta[0,\infty)-[0,\infty)$ *contains a copy of* M^u *for some* $u\in\omega^*$;

(2) There is no non-trivial simple point in Laver's model for Borel conjecture.

The second answers a question posed by Baldwin and Smith[1] negatively.

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§0. **Introduction.** In this paper, we study subcontinua of the Stone-Cech compactification of the reals. We refer to [7] and [11] for background on this topics. Let $M=\oplus_{n\in\omega} I_n$ be the topological sum of countably many copies of the unit interval. For any ultrafilter $u\in\omega^*$, we let $M^u=\bigcap\{cl_{\beta M}(\bigcup\{I_n: n\in A\}):A\in u\}$. It is not difficult to prove that M^u is a continuum (See, for example, [4]). If we let $i:M\longrightarrow\omega$ be the map defined by $i(r)=n$ for any $r\in I_n$ and $\beta i:\beta M\longrightarrow\beta\omega$ be the extension of i, it is easy to see that $M^u=\beta i^{-1}(u)$. So every subcontinuum of $\beta M-M$, therefore, every proper subcontinuum of $\beta[0,\infty)-[0,\infty)$, can be embedded into M^u for some $u\in\omega^*$. Moreover, we have

Theorem 1. *Every nondegenerate subcontinua of* $\beta[0,\infty)-[0,\infty)$ *contains a copy of* M^u *for some* $u\in\omega^*$.

For any map $f\in{}^{\omega}I$ and $u\in\omega^*$, let $f^u=\{F\subset M$: F is closed and $\{n:f(n)\in F\cap I_n\}\in u\}$ and $P^u=\{f^u:f\in{}^{\omega}I\}$. It is well known that f^u is a cut point of M^u if $\{n\in\omega:f(n)\neq 0,1\}\in u$ ((1) in [7]). It is also well known that there are many indecomposable subcontinua with cardinalities 2^c in M^u for any $u\in\omega^*$((19) in [7]). Therefore, by our Theorem 1, we have

Corollary. (a) *Every subcontinuum of* $\beta[0,\infty)-[0,\infty)$ *contains an indecomposable subcontinuum;*

(b) $\beta[0,\infty)$ *does not contain non-degenerate hereditarily indecomposable subcontinuum.*

(a) is due to D. P. Bellamy [2]. (b) was proved by M. Smith in [9] (van Douwen also announced it in [3]). The following problem was first posed by van Douwen (See the remarks at the end of [10]).

Question 1. (van Douwen) *Is there any cut point of* M^u *which is not in* P^u?

Definition 1. *A point* $x \in \beta M$ *is said to be (non-trivial) simple if for any* $F \in x$ *there is* $U \in x$ *such that* $U \subset F$ *and* $U \cap I_n = \phi$ *or* $U \cap I_n$ *is a (non-degenerate) interval.*

Fact 1. (a)(Corollary in §1 of [11]) *If* x *is a cut point of* M^u *and* $x \notin P^u$, *then* x *is a far point of* βM;

(b)(Theorem 1.1 in [11]) $x \in M^u$ *is a non-trivial simple if and only if* x *is a cut point of* M^u *and remote point of* βM.

The author [11] proved under CH that there is $u \in \omega^*$ such that there is a cut point of M^u which is not simple. Badlwin and Smith [1] proved that $MA_{\text{countable}}$ implies that there is a non-trivial simple point. They asked

Question 2.(Baldwin and Smith [1]) *Is there any non-trivial simple point in* ZFC ?

Theorem 2. *There is no non-trivial simple point in Laver's model* *for Borel conjecture.*

Question 1 remains open !

§1. Proof of Theorem 1. Let $X=[0,\infty)$ and $K\subset\beta X-X$ be a non-degenerate subcontinuum. The following lemma was proved by M. Smith in [9] for locally compact, locally connected metric spaces. We give a direct proof here.

Lemma 1.1. *Let* $\{U_0,U_1,\dots,U_m\}$ *be a finite open cover of* K *in* βX *such that* $U_i\cap K\neq\phi$ *for any* $i\leq m$. *Then there is a closed interval* $H\subset X$ *such that* $H\cap U_i\neq\phi$ *for* $i\leq m$ *and* $H\subset\cup\{U_i:i\leq m\}$.

Proof. Let $V=\bigcup\{U_0,U_1,\dots,U_m\}$ and $V'=V\cap X$. Then there are disjoint open intervals $\{J_n:n\in\omega\}$ so that $V'=\bigcup\{J_n:n\in\omega\}$. Let $A_0=\{n\in\omega:J_n\cap U_0\neq\phi\}$, $V_0=\bigcup\{J_n:n\in A_0\}$ and $W_0=\bigcup\{J_n:n\notin A_0\}$. We have $K\subset W\subset(\mathrm{cl}_{\beta X}V_0)\cup(\mathrm{cl}_{\beta X}W_0)$ and $(\mathrm{cl}_{\beta X}V_0)\cap(\mathrm{cl}_{\beta X}W_0)\subset(\mathrm{cl}_{\beta X}\bar{V}_0)\cap(\mathrm{cl}_{\beta X}\bar{W}_0)=\mathrm{cl}_{\beta X}(\bar{V}_0\cap\bar{W}_0)$, where $\bar{V}_0$ and $\bar{W}_0$ are the closures of V_0 and W_0 in X respectively. Since V is an open neighbourhood of K, we have $K\cap(\mathrm{cl}_{\beta X}(\bar{V}_0\cap\bar{W}_0))=\phi$. Therefore, $K\subset\mathrm{cl}_{\beta X}V_0$ since K is connected and $K\cap(\mathrm{cl}_{\beta X}V_0)\supset K\cap U_0\neq\phi$.

If we let $A_i=\{n\in\omega:J_n\cap U_j\neq\phi$ for $j\leq i\}$ and $V_i=\bigcup\{J_n:n\in A_i\}$ for $i\leq m$, we can easily show by induction that $K\subset\mathrm{cl}_{\beta X}V_i$ for $i\leq m$. So $A_m\neq\phi$. This completes the proof of Lemma 1.1.

We take U_0 and U_1 be disjoint open sets of βX so that $(\mathrm{cl}_{\beta X}U_0)\cap(\mathrm{cl}_{\beta X}U_1)=\phi$ and $U_i\cap K\neq\phi\,(i=0,1)$. Let $\mathscr{S}$ be the collection of closed intervals so that an interval $[a,b]$ belongs to $\mathcal{J}$ if and only if the following conditions hold:

(1) $[a,b]\cap(U_0\cup U_1)=\phi$ and $a\neq b$;

(2) $\{a,b\}\subset Br(U_0\cap X)\cup Br(U_1\cap X)$ and $a\in Br(U_0\cap X)$ if and only if $b\in Br(U_1\cap X)$,

where Br denotes the boundary operation in X. Since $cl_{\beta X}U_0$ and $cl_{\beta X}U_1$ are disjoint, $\mathcal{J}$ is discrete. We enumerate $\mathcal{J}$ as $\{J_n:n\in\omega\}$. We need only to show that there is $u\in\omega^*$ such that $\bigcap\{cl_{\beta X}(\cup\{J_n:n\in A\}):A\in u\}\subset K$. Let $\mathfrak{U}$ be an open neighbourhood base of K in βX. For $U\in\mathfrak{U}$, we let

$$A_U=\{n\in\omega:J_n\subset U\}.$$

By Lemma 1.1, we have $A_U\neq\phi$ for $U\in\mathfrak{U}$. Since $A_U\subset A_V$ for $U\subset V$ and $U,V\in\mathfrak{U}$, $\{A_U:U\in\mathfrak{U}\}$ has finite intersection property. Let

$$M_{\mathfrak{U}}=\bigcap\{cl_{\beta X}(\cup\{J_n:n\in A_U\}):U\in\mathfrak{U}\}.$$

Then $M_{\mathfrak{U}}\subset K$. For, if $x\in M_{\mathfrak{U}}\setminus K$, there is $U\in\mathfrak{U}$ such that $x\notin cl_{\beta X}U$. But $x\in M_{\mathfrak{U}}\subset cl_{\beta X}U$. Note that $cl_{\beta X}(\cup\{J_i:i\leq n\})\cap K=\phi$ for $n\in\omega$. So if u is an ultrafilter on ω and $\{A_U:U\in\mathfrak{U}\}\subset u$, then $u\in\omega^*$ and

$$\bigcap\{cl_{\beta X}(\cup\{J_n:n\in A\}):A\in u\}\subset K.$$

This completes the proof of our Theorem 1.

§2. Proof of Theorem 2. Recall that there is a natural partial order $<_u$ on M^u for $u\in\omega^*$ defined as follows: $x<_u y$ if and only if there are $F\in x$ and $H\in y$ such that $\{n\in\omega:F\cap I_n<H\cap I_n\}\in u$, where $F\cap I_n<H\cap I_n$ means that $r<s$ for any $r\in F\cap I_n$ and $s\in H\cap I_n$. It is easily seen that $(P^u,<_u)$ is isomorphic to the ultrapower $({}^{\omega}I/u,<_u)$. We consider the relation $\sim$ on M^u defined by $x\sim y$ if and only if $x=y$ or $x\not<y$ and $y\not<x$. It is very easy to

verify that ~ is an equivalence relation. A ~ equivalence class i.e., a maximal pairwise incomparable subset of $(M^u,<_u)$, is called a layer (this definition of layers is equivalent to Mioduszewski's original one in [7], see Lemma 1.2 in [11]). It can be proved easily from Mioduszewski's [7] that if x is a cut point of M^u, $\{x\}$ is a layer (Lemma1.3 in [11]). For any $A\subset{}^{\omega}I$ and $u\in\omega^*$, we let $A^u=\{f^u\in P^u : f\in A\}$. We say a pair $\mathscr{G}=(A,B)$ of subsets of ${}^{\omega}I$ determines a layer L in M^u for some $u\in\omega^*$ if the following two conditions hold:

(1) $A^u<_u B^u$, i.e., $f^u<_u g^u$ for any $f\in A$ and $g\in B$;

(2) for any $x\in M^u$, $x\in L$ if and only if $f^u<_u x<_u g^u$ for any $f\in A$ and $g\in B$.

If $L=\{x\}$ is a one point layer, we also say that x is determined by $\mathscr{G}$. Note that every layer is determined by a pair of subsets of ${}^{\omega}I$ (See [11], where we say layers are determined by gaps in $({}^{\omega}I\setminus u,<_u)$).

Let $\mathscr{P}=\bigcup_{n\in\omega}\mathscr{P}_n$ be a collection of closed rational subintervals of the unit interval I such that $\mathscr{P}_n$ is finite, pairwise disjoint and for any interval $J\subset I$, if the length of J is larger than $1/n$, then $|\{H\in\mathscr{P}_n : H\subset J\}|>n$. The following lemma is essentially Proposition 3.1 in [11].

Lemma 2.1. *Let* $\mathscr{G}=(A,B)$ *be a pair of of subsets of* ${}^{\omega}I$ *and* $A^u<_u B^u$ *for some* $u\in\omega^*$. $\mathscr{G}$ *determines a one point layer in* M^u *if and only if for any* $h\in{}^{\omega}\omega$, *there are* $f\in A$ *and* $g\in B$ *such that*

$$\{n\in\omega : \text{there is at most one } J\in\mathscr{P}_{h(n)}\ \ J\subset[f(n),g(n)]\}\in u.$$

By Lemma 2.1, we easily get

Lemma 2.2. *Let* $\mathfrak{M}\subset\mathfrak{N}$ *be models of* ZFC *such that there is* $r\in{}^{\omega}\omega\cap\mathfrak{N}$ *dominating every* $h\in{}^{\omega}\omega\cap\mathfrak{M}$ i.e, $h(n)<r(n)$ *for all but finitely many* $n\in\omega$. *Then no one point layer in* $\mathfrak{N}$ *is determined by a pair of subsets of* ${}^{\omega}I$ *in* $\mathfrak{M}$.

Let $\mathbb{P}_{\omega_2}$ be the ω_2 iteration of Laver forcing with countable support and $\mathbb{G}_{\omega_2}$ $\mathbb{P}_{\omega_2}$-generic over V. We assume that the continuum hypothesis holds in V. It is well-known that Laver real dominates every real in the ground model. Therefore, by Lemma 5.10 in [8] and Lemma 11 in [5], we have

Corollary 2.1. *There is no cut point in* M^u *determined by a pair of subsets of* ${}^{\omega}I$ *with cardinalities* ω_1 *in* $V[\mathbb{G}_{\omega_2}]$ *for any* $u\in\omega^*$.

The following lemma can be proved by modifying Miller's argument for Mathias forcing in §6 [6].

Lemma 2.3. *Suppose that* $p\Vdash_{\mathbb{P}_{\omega_2}}$ "$f:\omega\longrightarrow I$". *There are an extension* q *of* p *and a sequence* $\{c_n:n\in\omega\}$ *of codes for closed nowhere dense set in* V *such that* $q\Vdash_{\mathbb{P}_{\omega_2}}$ "$f(n)$ belongs to the set coded by c_n for $n\in\omega$".

Since every non-trivial simple point is a remote point of βM, we can easily see

Corollary 2.2. *Let* $x \in M^u$ *be a non-trivial simple point and* $\mathscr{E}=(A,B)$ *a pair subsets of* ${}^{\omega}I$ *determining* x. *Then in* $V[\mathbb{G}_{\omega_2}]$, *for any* $u' \in \omega^*$ *and* $u \subset u'$, *there is no* $f \in {}^{\omega}I$ *such that* $[A]_{u'} < [f] < [B]_{u'}$ *in* $({}^{\omega}I/u', <_{u'})$.

Now we are in a position to complete the proof of Theorem 2. Suppose that there is a non-trivial point $x \in M^u$ in $V[\mathbb{G}_{\omega_2}]$. Then there is a pair $\mathscr{E}=(A,B)$ of subsets of ${}^{\omega}I$ determining x. By Lemma 5.10 in [8], there is $\alpha < \omega_2$. such that in $V[\mathbb{G}_{\alpha}]$, x' is a non-trivial point of $M^{u'}$ and $\mathscr{E}'=(A',B')$ determines x', where $x'=x \cap V[\mathbb{G}_{\alpha}]$, $u'=u \cap V[\mathbb{G}_{\alpha}]$, $A'=A \cap V[\mathbb{G}_{a}]$ and $B'=B \cap V[\mathbb{G}_{\alpha}]$. By Lemma 11 in [5] and Corollary 2.2, $\mathscr{E}'$ determines x in $V[\mathbb{G}_{\omega_2}]$. This is impossible by Lemma 2.2.

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